Errors, Uncertainty & Statistics in Astronomy

Probably the most important skill for the practical astronomer is to be able to estimate the noise/uncertainty in their planned observations, and determine if the observations will meet their requirements.

Let us start with an example case that will illustrate the essence of getting the uncertainty estimation correct in astronomy. The example is of the well known "Hubble tension" in modern day cosmology. "Hubble's constant (H_o)" is arguable the most important number in cosmology, and represents how fast the Universe is expanding. Different groups of astronomers measured this using two different and independent methods.

1. From late Universe measurements of Type1a supernovae, $H_o = 74.22 \pm 1.82\ kms^{-1}Mpc^{-1}$.
2. From early Universe measurements by Planck satellite applied with the Λ-CDM model, $H_o = 67.4 \pm 0.5\ kms^{-1}Mpc^{-1}$.

These individual measurements are more than 3.5-σ away from each other, meaning that the chances that they are consistent with each other is less than 0.05%. So, either one or both of them have got their uncertainties wrong.

Astronomy, in essence, is an observational (not experimental) science. That is, observations of scientific objects is not done in controlled environments. In general, time on telescopes are heavily oversubscribed and we have just sufficient time that allows for obtaining marginal detection of sources, and we are often working at the very limits of detectability. So two things are of utmost importance in analysis: signal and noise. A good astronomer needs to be able to separate and quantify them.

Understanding and quantifying possible sources of errors is one of the most critical parts of doing science. The conclusions (and their strengths) that we draw from the data depends on how well we can control and understand the uncertainties.

1 Introduction to Uncertainties

Error (E) is defined as the difference between the TRUE value (x_o) and the measured value (x_i). That is, E = x_o - x_i. Most often, x_o is not generally not known, so is E. However, the likely errors can be "estimated", and these are called *"uncertainties"*. In other words, *"uncertainties"* are a quantification of the possible range of errors. So if we write the speed of car as x = 100 $ms^{-1} \pm 10\ ms^{-1}$, 10 ms^{-1} in our estimated uncertainty.

Every scientific measurement has an uncertainty in the "measured value"; no measurement is free of it. That is, even if external conditions such as temperature remains same and we repeat the measurement multiple times, the observed value will bounce around.

Figure 1 shows such a typical distribution of measurements that is spread around in the form of a Gaussian/Normal distribution. As will be discussed later, the probability distribution of uncertainties is almost always approximated by a Normal distribution.

In general, there are two kinds of uncertainties:

1. *Systematic Uncertainty*: There are due to faults/bias in our measurement method. As a consequence, these get repeated in all measurements. The good thing is that because these

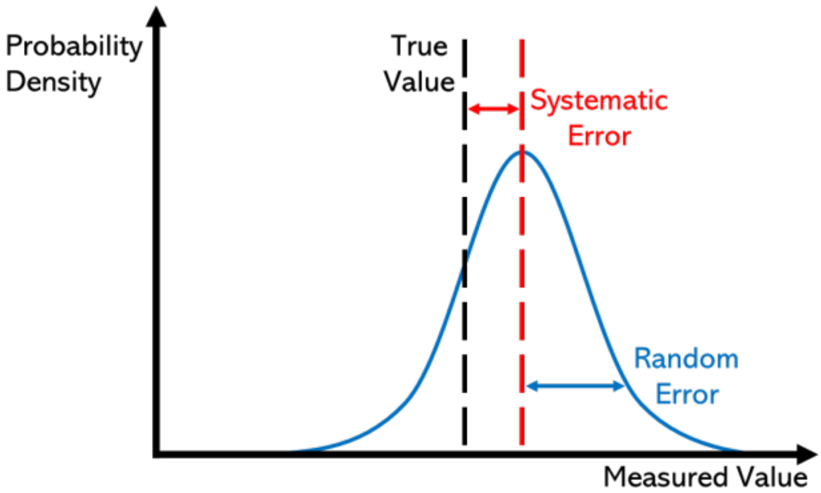

Figure 1: A plot showing the distribution of measurements taken for a parameter showing the effect of systematic error and normally distributed random error. Source: Wikipedia.

get repeated, they can often be reliably measured and corrected. A good experimenter gets rid of "all" systematic errors. These are due to imperfect equipment, improper or biased observation or presence of additional physical effects (eg. friction, extinction of the atmosphere and ISM) that are not taken into account. An example of a systematic error is that due to temperature, the length of a scale is smaller than it should be.

Question: If you have a protractor, will its measurements of reading be dependent on temperature.

2. *Random Uncertainty*: By nature, these uncertainties are random and their source is not often known. It can be a combination of temperature, vibrations, humidity, electronic noise etc etc. These follow Gaussian/normal distribution. It is also called "statistical uncertainty". These are unavoidable. Because they follow a Normal distribution (a consequence of Central Limit Theorem), they will spread around a mean value and will be unbiased.

   A very important aspect of measurement dominated by random uncertainties are that repeating the measurements averaging reduces the error.

Precision and accuracy are independent of each other. Eg. shooting of a basketball.
the best way to estimate uncertainity is by taking the standard deviation, σ.

- Accuracy. Accuracy is an estimate/measure of the closenss of the measured value to the true value. In the above example, 10 m/s is our estimated accuracy
- Precision: Estimate of the repeatability of the measurements, "not" necessarily true value.
- Accuracy and sensitivity are not the same.

  Accuracy $\neq$ Sensitivity

1.1 How find uncertainty

Looking at Figure 1, one can think how to estimate the systematic and random uncertainties. If we make a measurement of a quantity x multiple times, the spread of the measurements provides us the measure of random error. But we cannot know if and how much of systematic error is

(a) High accuracy and low precision. (b) Low accuracy and high precision.

Figure 2: Illustration of how experiments can have high accurate and low precision and vice versa.

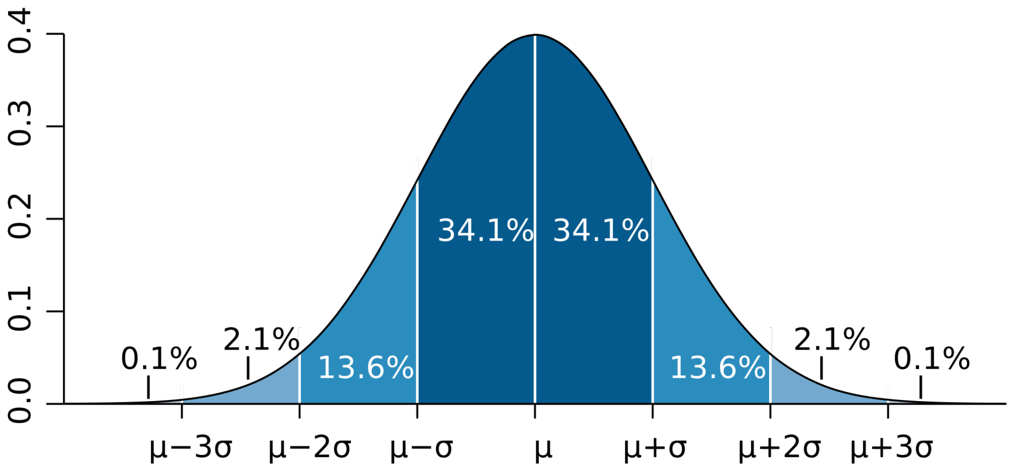

Figure 3: Probability distribution of a normal function with mean and variance given by μ and σ^2.

present in our measurements. For that, we need to measure standard sources/objects whose True value is known apriori. For example, to estimate the "systematic" uncertainties in the flux measurement of astronomy cameras, we use standard photometric stars whose flux is known and is stable.

The standard deviation is the measure of the random uncertainty.

$$\bar{x} = \frac{\sum_{j=1}^{n} x_j}{n} \tag{1}$$

$$\sigma^2 = \frac{\sum_{j=1}^{n} (\bar{x} - x_j)^2}{n-1} \tag{2}$$

The uncertainty in the measurement of x, Δx is given by

$$\Delta\bar{x} = \frac{\bar{x}}{\sqrt{n}} \tag{3}$$

Uncertainty in the Mean $(\Delta\bar{x})$Therefore, the more measurements we do, the lesser is the uncertainty. So, the standard deviation gives the uncertainty for a single measurement (i.e of the measurement system and method), while the $\Delta\bar{x}$ is the uncertainty on the mean.

The measured value is given as:

$$x_m = \bar{x} \pm \frac{\sigma}{\sqrt{n}} \tag{4}$$

Standard deviation (σ) is important for determining uncertainties. For a Gaussian distribution:

No of photons (N)	Noise	SNR
1	1	1
10	$\sqrt{10}$	$\sqrt{10}$
100	10	10
10^3	$10\sqrt{10}$	$10\sqrt{10}$
10^4	100	100

Table 1: Why do astronomers want to build larger and larger telescopes? The more photons we collect, the associated uncertainty increases, but the order of increase in signal is more.

- 68.3 % probability that the true values is within 1 σ of the mean.
- 95.4 % probability that the true values is within 2 σ of the mean.
- 99.7 % probability that the true values is within 3 σ of the mean.

1.2 Photon noise & Poisson Distribution

Measurement of photons, like all counting processes, follows Poisson distribution. If a source is emitting N photons per unit time , it does not mean we will receive N photons in every unit time interval. In fact, in every unit time interval, we will receive a variable number of photons in our detector. These variations are called "photon/shot" noise. It represents the irreducible minimum level of uncertainty present in astronomical measurements, even in the most ideal scenario. The probability of detecting an x number of photons is given by Equation 5.

- An example would be how many emails you recieve per day.

$$p(x) = \frac{N^x e^{-N}}{x!} \tag{5}$$

- One very useful and nice aspect of Poisson distribution is that it is very well approximated by a Gaussian distribution when $N > 5$, which is always the case in astronomy as we deal with large N.

One of the key features of the Poisson distribution is the variance (σ^2) and mean value is λ. Therefore, if we detect N photons in our detector from a source, the best we can do is to say that the true mean photon rate from the star has a likely mean of N, and an uncertainty of $\sqrt{N}$.

- It is important to note that the above conclusion applies not only to photons from stars, but to any counting process such as the counting of photons from the sky (i.e sky background), or the production of thermally-generated electrons in a semi-conductor (i.e. dark current).

So, for an ideal detector in space, the signal to noise ratio would be just $\sqrt{N}$, as shown below.

$$\sigma = \sqrt{N} \tag{6}$$

$$SNR = \frac{N}{\sqrt{N}} = \sqrt{N} \tag{7}$$

As can be seen, while the signal increases linearly with the number of photons we collect, so does the noise, but with a smaller power. This is the reason astronomers strive to collect more light by building these huge telescopes. Table 1 shows how the SNR progresses in this ideal scenario.

1.3 Example: Calculating SNR and uncertainty in CCD photometry

For CCD observations there are several sources of noise. Suppose the number of photo-electrons detected from the object, sky background and dark current are N_o, N_s and N_d, respectively. The major source of noise in CCD observations and their associated uncertainties are as follows:

1. shot-noise in the detected photo-electrons from the source, $\sqrt{N_o}$;
2. shot-noise in the detected photo-electrons from the sky background, $\sqrt{N_s}$;
3. shot-noise in the thermally excited electrons, i.e the dark current, $\sqrt{N_d}$;
4. time-independent readout noise, R. Note there is no square root here. The readout noise is the standard deviation in the number of electrons measure - it is not a Poissonian counting process.

Sum of multiple Gaussian distributions of mean and standard deviations is sum of all variances.

$$\sigma_T^2 = \sum_{j=1}^{n} \sigma_i^2 \tag{8}$$

Therefore, the total noise, σ_T is given by:

$$\sigma_T = \sqrt{N_o + N_s + N_d + R^2} \tag{9}$$

Let S_o, S_s and S_d be the average rate of arrival/accumulation of the photons/electrons and Q is the quantum efficiency of the detector. The dependence of SNR on all the above and the telescope integration time is given by the following equations.

$$SNR = \frac{N_o}{\sigma_T} = \frac{N_o}{\sqrt{N_o + N_s + N_d + R^2}} \tag{10}$$

$$SNR = \frac{N_o}{\sigma_T} = \frac{S_oQt}{\sqrt{S_oQt + S_sQtn_p + S_dtn_p + R_o^2n_p}} \tag{11}$$

Limiting cases:

1. Object Limited. If S_o is much greater than S_s, S_d and R^2, then

$$SNR = \frac{S_oQt}{\sqrt{S_oQt}} = \sqrt{S_oQt} \tag{12}$$

   Therefore, SNR $\propto \sqrt{t}$

2. Background Limited.

$$SNR = \frac{S_oQt}{\sqrt{S_sQtn_p}} = \frac{S_o\sqrt{Qt}}{\sqrt{S_sn_p}} \propto \frac{S_o\sqrt{t}}{\sqrt{S_s}} \tag{13}$$

   For a fixed S_o, the SNR scales with the square root of the sky signal N_s. It is therefore very important to observe faint sources when the sky is faint. Scattered light from the from the Moon is a major source of sky background, so faint sources should be observed near new Moon.

3. Read Noise Limited.

$$SNR = \frac{S_oQt}{R_o\sqrt{n_p}} = \frac{N_o}{R} \tag{14}$$

   So the SNR is linearly dependent on the integration time as R_o is independant of integration time or the telescope aperture.

2 Single Variate Function Error Analysis

If $f(x)$ is a function of x, and we know the mean value and error in measurement of x, we can find the mean value and error in the value of $f(x)$ using Equations 15 and 16 ($f'(\bar{x}) = \frac{\partial f(\bar{x})}{\partial x}$).

$$f(\bar{x}) = \bar{f} = f(\bar{x}) \tag{15}$$

$$\sigma^2(f) = f'(\bar{x})^2 \sigma_x^2 \tag{16}$$

3 Multivariate Function Errors

Now lets take error propagation a step further: how does the error propagate in a function with two variables? Suppose we have a function $f(x, y)$ made from two variables x and y. x has been measured with mean value $\bar{x}$ and measurement uncertainty σ_x. Similarly, y has been measured with mean value $\bar{y}$ and measurement uncertainty σ_y. The mean and error of f(x,y) is given by Equations 17 and 18.

$$\bar{f}(x, y) = \bar{f} = f(\bar{x}, \bar{y}) \tag{17}$$

$$\sigma^2(f) = f'_x(\bar{x}, \bar{y})^2 \sigma_x^2 + f'_y(\bar{x}, \bar{y})^2 \sigma_y^2 \tag{18}$$

Here, $f'(\bar{x}) = \frac{\partial f(\bar{x}, \bar{y})}{\partial x}$ and, $f'(\bar{y}) = \frac{\partial f(\bar{x}, \bar{y})}{\partial y}$.

4 Polarization Error Analysis Formulae

$$q = \frac{N_0 - N_1}{N_0 + N_1} \tag{19}$$

$$u = \frac{N_2 - N_3}{N_2 + N_3} \tag{20}$$

If σ_{N0}, σ_{N1}, σ_{N2} and σ_{N3} are the errors in intensity measurements of N_0, N_1, N_2 and N_3, using Equation 18, the expression for errors in q and u are given by Equations 21 and 22.

$$\sigma_q = \frac{2}{(N_0 + N_1)^2} \sqrt{\sigma_{N0}^2 {N_1}^2 + \sigma_{N1}^2 {N_0}^2} \tag{21}$$

$$\sigma_u = \frac{2}{(N_2 + N_3)^2} \sqrt{\sigma_{N2}^2 {N_3}^2 + \sigma_{N3}^2 {N_2}^2} \tag{22}$$

We know, $p = f(q, u)$ and $\theta = f(q, u)$ as given by the equations below.

$$p = \sqrt{q^2 + u^2} \tag{23}$$

$$\theta = \frac{1}{2} \tan^{-1}(\frac{u}{q}) \tag{24}$$

The uncertainty in measurements of q and u gets propagated to uncertainties in p and θ measurements as per Equations 25 and 26.

$$\sigma_p = \sqrt{\frac{q^2 \sigma_q^2 + u^2 \sigma_u^2}{q^2 + u^2}} \tag{25}$$

$$\sigma_\theta = \frac{1}{2}\sqrt{\frac{q^2\sigma_u^2 + u^2\sigma_q^2}{(q^2+u^2)^2}} \tag{26}$$

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5 References

1. Mathematical Methods for Physicists by Arfken & Weber.
2. https://www.physics.columbia.edu/sites/default/files/content/Lab%20Resources/Lab%20Guide%201_%20Introduction%20to%20Error%20and%20Uncertainty.pdf
3. https://slittlefair.staff.shef.ac.uk/teaching/phy241/lectures/l09/
4. https://www.physics.upenn.edu/sites/default/files/Managing%20Errors%20and%20Uncertai pdf
5. https://sphweb.bumc.bu.edu/otlt/mph-modules/bs/bs704_probability/BS704_Probability1 html